Fundamentals of Rock Mechanics

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Fourth Edition

J. C. Jaeger, N. G. W. Cook, and R. W. Zimmerman

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Preface to the Fourth Edition

When the first edition of this book appeared in 1969, rock mechanics had only recently begun to emerge as a distinct and identifiable scientific subject. It coalesced from several strands, including classical continuum mechanics, engineering and structural geology, and mining engineering. The two senior authors of *Fundamentals of Rock Mechanics* were perhaps uniquely qualified to play seminal roles in bringing about this emergence. John Jaeger had by that time already enjoyed a long and distinguished career as arguably the preeminent applied mathematician of the English-speaking world, and was the coauthor, with H. S. Carslaw, of one of the true classics of the scientific literature, *Conduction of Heat in Solids*. Neville Cook was at that time barely 30 years old, but was already the director of research at the South African Chamber of Mines, and well on his way to becoming acknowledged as the leading and most brilliant figure in this new field of rock mechanics.

The earlier editions of this book played a large role in establishing an identity for the field of rock mechanics and in defining what are now accepted to be the "fundamentals" of the field. These fundamentals consist firstly of the classical topics of solid mechanics – stress and strain, linear elasticity, plasticity, viscoelasticity, and elastic wave propagation. But rocks are much more complex than are most of the traditional engineering materials for which the classical mechanics theories were intended to apply. Hence, a book entitled *Fundamentals of Rock Mechanics* must also treat certain topics that are either unique to rocks, or at any rate which assume great importance for rocks, such as friction along rough surfaces, degradation and failure under compressive loads, coupling between mechanical deformation and fluid flow, the effect of cracks and pores on mechanical deformation, and, perhaps most importantly, the effect of fractures and joints on large-scale rock behavior.

Rock mechanics, thus defined, forms a cornerstone of several fields of science and engineering – from structural geology and tectonophysics, to mining, civil, and petroleum engineering. A search of citations in scientific journals shows that previous editions of this book have found an audience that encompasses not only these areas, but also includes material scientists and ceramicists, for example. It is hoped that this new edition will continue to be found useful by such a variety of researchers, students, and practitioners.

The extent to which the different chapters of this edition are new or expanded varies considerably, but aside from the brief, introductory Chapter 1, all have been revised and updated to one extent or another. The discussion of the basic theory of stress and strain in Chapter 2 has now been complemented by extensive use of vector and matrix notation, although all of the major results are also displayed in explicit component form. A discussion of rate-dependence has been added to Chapter 3 on friction. Chapter 4 on rock deformation has been updated, with more emphasis on true-triaxial failure criteria. Chapter 5 on linear elasticity now includes more discussion of anisotropic elasticity, as well as coverage of important general theorems related to strain energy. A detailed discussion of issues related to measurement of the strain-softening portion of the complete stress–strain curve has been added to Chapter 6 on laboratory measurements. Chapter 7 on poroelasticity is almost entirely new, and also includes a new section on thermoelasticity. Chapter 8 on stresses around cavities and inclusions, which is based heavily on the chapter in the 3rd edition that was entitled "Further Problems in Elasticity," has been simplified by moving some material to other more appropriate chapters, while at the same time adding material on three-dimensional problems. The chapters of the 3rd edition on ductile materials, granular materials, and time-dependent behavior have been combined to form Chapter 9 on inelastic behavior. Chapter 10, on micromechanical models, is a greatly enlarged and updated version of the old chapter on crack phenomena, with expanded treatment of effective medium theories. Chapter 11 on wave propagation has been doubled in size, with new material on reflection and refraction of waves across interfaces, the effects of pore fluids, and attenuation mechanisms. The important influence of rock fractures on the mechanical, hydraulic, and seismic behavior of rock masses is now widely recognized, and an entirely new chapter, Chapter 12, has been devoted to this topic. Chapter 13 on subsurface stresses collects material that had been scattered in various places in the previous editions. The final chapter, Chapter 14, briefly shows how the ideas and results of previous chapters can be used to shed light on some important geological and geophysical phenomena.

In keeping with the emphasis on fundamentals, this book contains no discussion of computational methods. Methods such as boundary elements, finite elements, and discrete elements are nowadays an indispensable tool for analyzing stresses and deformations around subsurface excavations, mines, boreholes, etc., and are also increasingly being used to study problems in structural geology and tectonophysics. But the strength of numerical methods has, at least until now, been in analyzing specific problems involving complex geometries and complicated constitutive behavior. Analytical solutions, although usually limited to simplified geometries, have the virtue of displaying the effect of the parameters of a problem, such as the elastic moduli or crack size, in a clear and transparent way. Consequently, many important analytical solutions are derived and/or presented in this book.

The heterogeneous nature of rock implies that most rock properties vary widely within a given rock type, and often within the same reservoir or quarry. Hence, rock data are presented in this work not to provide "handbook values" that could be used in specific applications, but mainly to illustrate trends, or to highlight the level of agreement with various models and theories. Nevertheless, this new edition contains slightly more actual rock data than did the previous edition, as measured by the number of graphs and tables that contain laboratory or field data. The reference list contains about 15% more items than in the 3rd edition, and more than half of the references are new. With only a few exceptions for some key references that originally appeared in conference proceedings or as institutional reports or theses, the vast majority of the references are to journal articles or monographs.

The ordering of the chapters remains substantially the same as in the 3rd edition. The guiding principle has been to minimize, as much as possible – in fact, almost entirely – the need to refer in one chapter to definitions, data or theoretical results that are not presented until a later chapter. In particular, then, the chapters are not structured so as to follow the workflow that would be used in a rock engineering project. For example, although knowledge of the *in situ* stresses would be required at the early stages of an engineering project, the chapter on subsurface stresses is placed near the end, because its presentation requires reference to analytical solutions that have been developed in several previous chapters.

The mathematical level of this edition is the same as in previous editions. The mathematical tools used are those that would typically be learned by undergraduates in engineering or the physical sciences. Thus, matrix methods are now extensively used in the discussion of stress and strain, as these have become a standard part of the undergraduate curriculum. Conversely, Cartesian tensor indicial notation, which is convenient for presenting the equations of stress, strain, and elasticity, has not been used, as it is not widely taught at undergraduate level. Perhaps the only exception to this rule is the use in Chapter 8 of functions of a complex variable for solving two-dimensional elasticity problems. But the small amount of complex variable theory that is required is presented as needed, and the integral theorems of complex analysis are avoided.

Rock mechanics is indeed a subfield of continuum mechanics, and my contribution to this book owes a heavy debt to the many excellent teachers of continuum mechanics and applied mathematics with whom I have been fortunate enough to study. These include Melvin Baron, Herbert Deresiewicz, and Morton Friedman at Columbia, and David Bogy, Michael Carroll, Werner Goldsmith, and Paul Naghdi at Berkeley. Although this book shows little obvious influence of Paul Naghdi's style of continuum mechanics, it was only after being inspired by his elegant and ruthlessly rigorous approach to this subject that I changed my academic major field to continuum mechanics, thus setting me on a path that led me to do my PhD in rock mechanics.

Finally, I offer my sincere thanks to John Hudson of Imperial College and Rock Engineering Consultants, and Laura Pyrak-Nolte of Purdue University for reading a draft of this book and providing many valuable suggestions.

R. W. Zimmerman
Stockholm, May 2006

Artwork from the book is available to instructors at:
www.blackwellpublishing.com/jaeger

To my wife, Jennifer, my partner in everything

Neville Cook,
Lafayette, Calif.
January 1998

1 Rock as a material

1.1 Introduction

Rock mechanics was defined by the Committee on Rock Mechanics of the Geological Society of America in the following terms: "Rock mechanics is the theoretical and applied science of the mechanical behavior of rock; it is that branch of mechanics concerned with the response of rock to the force fields of its physical environment" (Judd, 1964). For practical purposes, rock mechanics is mostly concerned with rock masses on the scale that appears in engineering and mining work, and so it might be regarded as the study of the properties and behavior of accessible rock masses due to changes in stresses or other conditions. Since these rocks may be weathered or fragmented, rock mechanics grades at one extreme into soil mechanics. On the other hand, at depths at which the rocks are no longer accessible to mining or drilling, it grades into the mechanical aspects of structural geology (Pollard and Fletcher, 2005).

Historically, rock mechanics has been very much influenced by these two subjects. For many years it was associated with soil mechanics at scientific conferences, and there is a similarity between much of the two theories and many of the problems. On the other hand, the demand from structural geologists for knowledge of the behavior of rocks under conditions that occur deep in the Earth's crust has stimulated much research at high pressures and temperatures, along with a great deal of study of the experimental deformation of both rocks and single crystals (Paterson and Wong, 2005).

An important feature of accessible rock masses is that they are broken up by joints and faults, and that pressurized fluid is frequently present both in open joints and in the pores of the rock itself. It also frequently happens that, because of the conditions controlling mining and the siting of structures in civil engineering, several lithological types may occur in any one investigation. Thus, from the outset, two distinct problems are always involved: (i) the study of the orientations and properties of the joints, and (ii) the study of the properties and fabric of the rock between the joints.

In any practical investigation in rock mechanics, the first stage is a geological and geophysical investigation to establish the lithologies and boundaries of the rock types involved. The second stage is to establish, by means of drilling or investigatory excavations, the detailed pattern of jointing, and to determine the mechanical and petrological properties of the rocks from samples. The third stage, in many cases, is to measure the *in situ* rock stresses that are present in the unexcavated rock. With this information, it should be possible to predict the response of the rock mass to excavation or loading.

This chapter presents a very brief introduction to the different rock types and the manner in which rock fabric and faulting influences the rock's engineering properties. A more thorough discussion of this topic can be found in Goodman (1993).

1.2 Joints and faults

Joints are by far the most common type of geological structure. They are defined as cracks or fractures in rock along which there has been little or no transverse displacement (Price, 1966). They usually occur in sets that are more or less parallel and regularly spaced. There are also usually several sets oriented in different directions, so that the rock mass is broken up into a blocky structure. This is a main reason for the importance of joints in rock mechanics: they divide a rock mass into different parts, and sliding can occur along the joint surfaces. These joints can also provide paths for fluids to flow through the rock mass.

Joints occur on all scales. Joints of the most important set, referred to as *major joints*, can usually be traced for tens or hundreds of meters, and are usually more or less planar and parallel to each other. The sets of joints that intersect major joints, known as *cross joints*, are usually of less importance, and are more likely to be curved and/or irregularly spaced. However, in some cases, the two sets of joints are of equal importance. The spacing between joints may vary from centimeters to decameters, although very closely spaced joints may be regarded as a property of the rock fabric itself.

Joints may be "filled" with various minerals, such as calcite, dolomite, quartz or clay minerals, or they may be "open," in which case they may be filled with fluids under pressure.

Jointing, as described above, is a phenomenon common to all rocks, sedimentary and igneous. A discussion of possible mechanisms by which jointing is produced is given by Price (1966) and Pollard and Aydin (1988). Joint systems are affected by lithological nature of the rock, and so the spacing and orientation of the joints may change with the change of rock type.

Another quite distinct type of jointing is *columnar jointing*, which is best developed in basalts and dolerites, but occasionally occurs in granites and some metamorphic rocks (Tomkeieff, 1940; Spry, 1961). This phenomenon is of some importance in rock mechanics, as igneous dykes and sheets are frequently encountered in mining and engineering practice. In rocks that have columnar jointing, the rock mass is divided into columns that are typically hexagonal, with side lengths on the order of a few tens of centimeters. The columns are intersected by cross joints that are less regular toward the interior of the body. The primary cause of columnar jointing appears to be tensile stresses that are created by thermal contraction during cooling. At an external surface, the columns run normal to the surface, and Jaeger (1961) and others have suggested that in the interior of the rock mass the columns run normal to the isotherms during cooling. The detailed mechanism of columnar jointing has been discussed by

Lachenbruch (1961); it has similarities to the cracks that form in soil and mud during drying, and to some extent to cracking in permafrost.

Faults are fracture surfaces on which a relative displacement has occurred transverse to the nominal plane of the fracture. They are usually unique structures, but a large number of them may be merged into a *fault zone*. They are usually approximately planar, and so they provide important planes on which sliding can take place. Joints and faults may have a common origin (de Sitter, 1956), and it is often observed underground that joints become more frequent as a fault is approached. Faults can be regarded as the equivalent, on a geological scale, of the laboratory shear fractures described in Chapter 4. The criteria for fracturing developed in Chapter 4 are applied to faults in §14.2.

From the point of view of rock mechanics, the importance of joints and faults is that they cause the existence of fairly regularly spaced, approximately plane surfaces, which separate blocks of "intact" rock that may slide on one another. In practice, the essential procedure is to measure the orientation of all joint planes and similar features, either in an exploratory tunnel or in a set of boreholes, and to plot the directions of their normal vectors on a stereological projection. Some typical examples are shown in the following figures taken from investigations of the Snowy Mountain Hydroelectric Authority in Australia.

Figure 1.1 is a stereographic projection plot of the normals to the fracture planes in the Headrace Channel for the Tumut 3 Project. The thick lines show the positions of the proposed slope cuts. In this case, 700 normal vectors were measured.

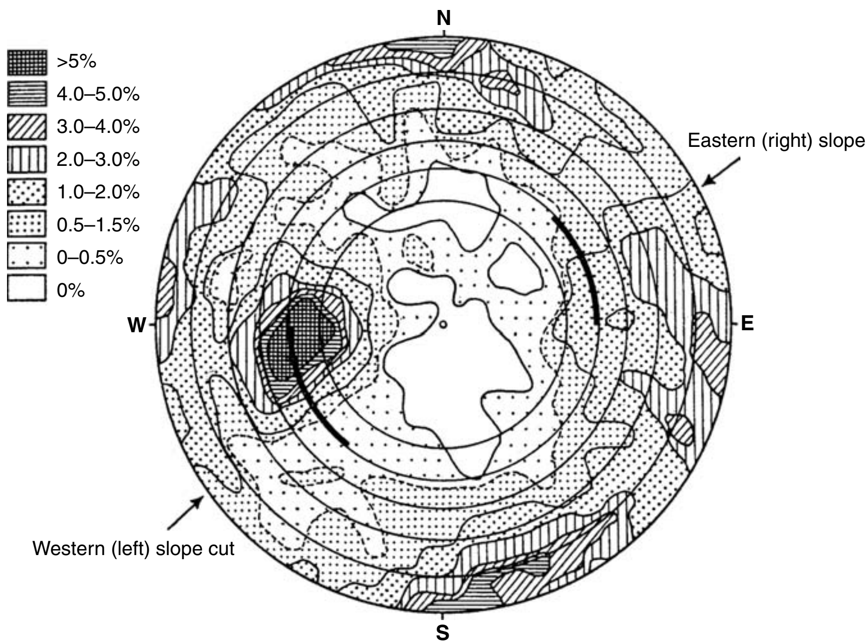

Fig. 1.1
Stereographic plot (lower hemisphere) of normals to fracture planes in Tumut 3 Headrace Channel. The contours enclose areas of equal density of poles.

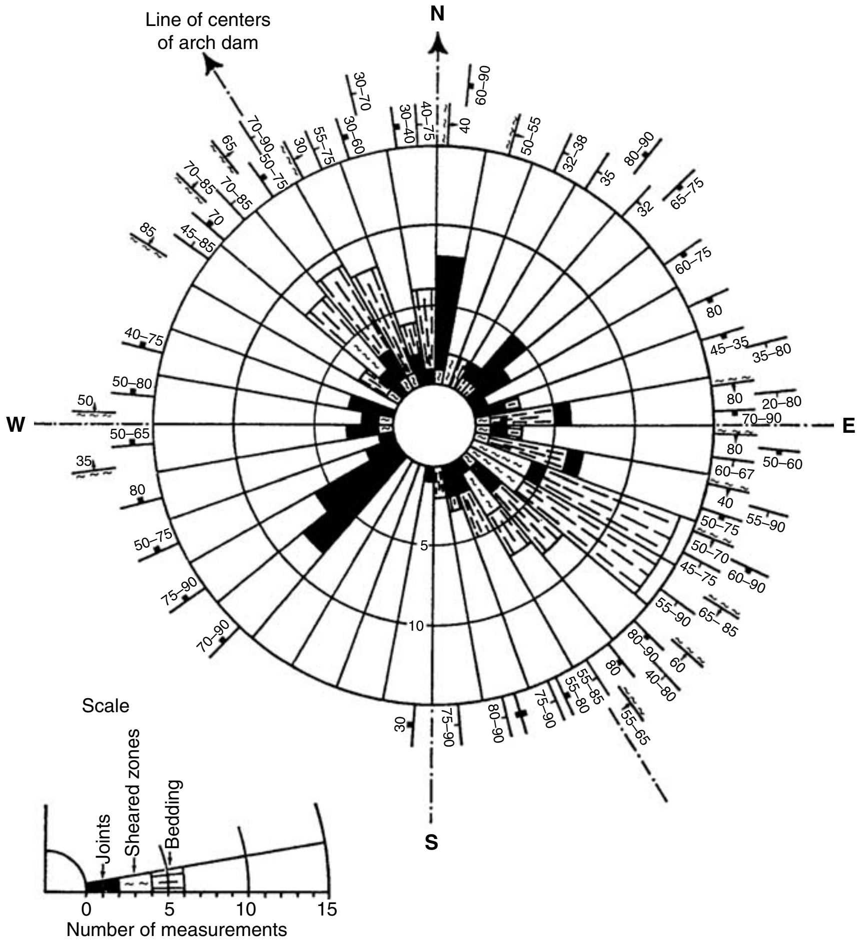

Fig. 1.2 Rosette diagram showing strikes of joints, sheared zones, and bedding planes at the Murray 2 dam site. The predominant dip for each strike is also shown.

Figure 1.2 shows the important geological features at the Murray 2 dam site on a different representation. Here, the directions of strike of various features are plotted as a rosette, with the angles of dip of the dominant features at each strike given numerically. The features recorded are joints, sheared zones, and bedding planes, any or all of which may be of importance.

Finally, Fig. 1.3 gives a simplified representation of the situation at the intersection of three important tunnels. There are three sets of joints whose dips and strikes are shown in Fig. 1.3.

1.3 Rock-forming minerals

Igneous rocks consist of a completely crystalline assemblage of minerals such as quartz, plagioclase, pyroxene, mica, etc. Sedimentary rocks consist of an assemblage of detrital particles and possibly pebbles from other rocks, in a matrix of materials such as clay minerals, calcite, quartz, etc. From their nature, sedimentary rocks contain voids or empty spaces, some of which may form an interconnected system of pores. Metamorphic rocks are produced by the action of heat, stress, or heated fluids on other rocks, sedimentary or igneous.

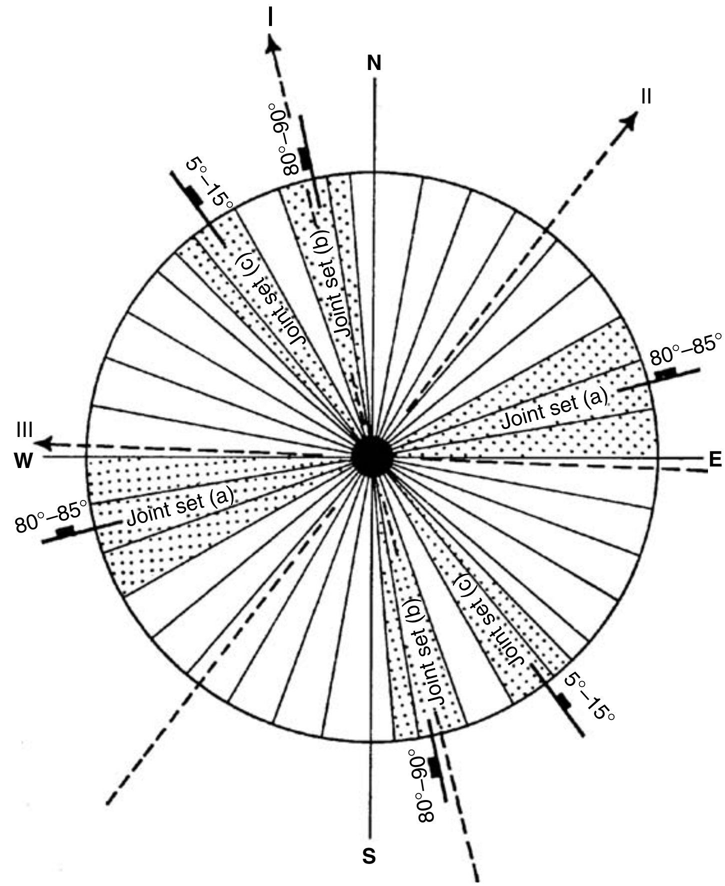

Fig. 1.3 Dips and strikes of three joint sets, (a) (b) and (c), at the intersection of three tunnels: I, Island Bend intake; II, Eucumbene-Snowy tunnel; III, Snowy-Geehi tunnel.

All of these minerals are anisotropic, and the elastic moduli of the more common ones, as defined in §5.10, are known numerically. If in a polycrystalline rock there are any preferred orientations of the crystals, this will lead to anisotropy of the rock itself. If the orientations of the crystals are random, the rock itself will be isotropic, and its elastic moduli may be estimated by the methods described in §10.2.

There are a number of general statistical correlations between the elasticity and strength of rocks and their petrography, and it is desirable to include a full petrographic description with all measurements. Grain size also has an effect on mechanical properties. In sedimentary rocks there are, as would be expected, some correlations between mechanical properties and porosity (Mavko et al., 1998).

A great amount of systematic research has been done on the mechanical properties of single crystals, both with regards to their elastic properties and their plastic deformation. Single crystals show preferred planes for slip and twinning, and these have been studied in great detail; for example, calcite (Turner et al., 1954) and dolomite (Handin and Fairbairn, 1955). Such measurements are an essential preliminary to the understanding of the fabric of deformed rocks, but they have little relevance to the macroscopic behavior of large polycrystalline specimens.

1.4 The fabric of rocks

The study of the fabric of rocks, the subject of *petrofabrics*, is described in many books (Turner and Weiss, 1963). All rocks have a fabric of some sort. Sedimentary rocks have a primary depositional fabric, of which the bedding is the most obvious element, but other elements may be produced by currents in the water. Superimposed on this primary fabric, and possibly obscuring it, may be fabrics determined by subsequent deformation, metamorphism, and recrystallization.

The study of petrofabrics comprises the study of all fabric elements, both microscopic and macroscopic, on all scales. From the present point of view, the study of the larger elements, faults and relatively widely spaced joints, is an essential part of rock mechanics. Microscopic elements and very closely spaced features such as cleats in coal, are regarded as determining the fabric of the rock elements between the joints. These produce an anisotropy in the elastic properties and strength of the rock elements. In principle, this anisotropy can be measured completely by mechanical experiments on rock samples, but petrofabric measurements can provide much useful information, in particular about preferred directions. Petrofabric measurements are also less time-consuming to make, and so are amenable to statistical analysis. Studies of rock fabric are therefore better made by a combination of mechanical and petrofabric measurements, but the latter cannot be used as a substitute for the former. Combination of the two methods has led to the use of what may be regarded as standard anisotropic rocks. For example, Yule marble, for which the calcite is known (Turner, 1949) to have a strong preferred orientation, has been used in a great many studies of rock deformation (Turner et al., 1956; Handin et al., 1960).

A second application of petrofabric measurements in rock mechanics arises from the fact that some easily measured fabric elements, such as twin lamellae in calcite and dolomite, quartz deformation lamellae, kink bands, and translation or twin gliding in some crystals, may be used to infer the directions of the principal stresses under which they were generated. These directions, of course, may not necessarily be the same as those presently existing, and so they form an interesting complement to underground stress measurements. Again, such measurements are relatively easy to make and to study statistically. The complete fabric study of joints and fractures on all scales is frequently used both to indicate the directions of the principal stresses and the large-scale fabric of the rock mass as a whole (Gresseth, 1964).

A great deal of experimental work has been concentrated on the study of the fabrics produced in rocks in the laboratory under conditions of high temperature and pressure. In some cases, rocks of known fabric are subjected to prescribed laboratory conditions, and the changes in the fabric are studied; for example, Turner et al. (1956) on Yule marble, and Friedman (1963) on sandstone.

Alternatively, specific attempts to produce certain types of fabrics have been made. Some examples are the work of Carter et al. (1964) on the deformation of quartz, Paterson and Weiss (1966) on kink bands, and Means and Paterson (1966) on the production of minerals with a preferred orientation.

Useful reviews of the application of petrofabrics to rock mechanics and engineering geology have been given by Friedman (1964) and Knopf (1957).

1.5 The mechanical nature of rock

The mechanical structure of rock presents several different appearances, depending upon the scale and the detail with which it is studied.

Most rocks comprise an aggregate of crystals and amorphous particles joined by varying amounts of cementing materials. The chemical composition of the crystals may be relatively homogeneous, as in some limestones, or very heterogeneous, as in a granite. Likewise, the size of the crystals may be uniform or variable, but they generally have dimensions of the order of centimeters or small fractions thereof. These crystals generally represent the smallest scale at which the mechanical properties are studied. On the one hand, the boundaries between crystals represent weaknesses in the structure of the rock, which can otherwise be regarded as continuous. On the other hand, the deformation of the crystals themselves provides interesting evidence concerning the deformation to which the rock has been subjected.

On a scale with dimensions ranging from a few meters to hundreds of meters, the structure of some rocks is continuous, but more often it is interrupted by cracks, joints, and bedding planes that separate different strata. It is this scale and these continuities which are of most concern in engineering, where structures founded upon or built within rock have similar dimensions.

The overall mechanical properties of rock depend upon each of its structural features. However, individual features have varying degrees of importance in different circumstances.

At some stage, it becomes necessary to attach numerical values to the mechanical properties of rock. These values are most readily obtained from laboratory measurements on rock specimens. These specimens usually have dimensions of centimeters, and contain a sufficient number of structural particles for them to be regarded as grossly homogeneous. Thus, although the properties of the individual particles in such a specimen may differ widely from one particle to another, and although the individual crystals themselves are often anisotropic, the crystals and the grain boundaries between them interact in a sufficiently random manner so as to imbue the specimen with average homogeneous properties. These average properties are not necessarily isotropic, because the processes of rock formation or alteration often align the structural particles so that their interaction is random with respect to size, composition and distribution, but not with respect to their anisotropy. Nevertheless, specimens of such rock have gross anisotropic properties that can be regarded as being homogeneous.

On a larger scale, the presence of cracks, joints, bedding and minor faulting raises an important question concerning the continuity of a rock mass. These disturbances may interrupt the continuity of the displacements in a rock mass if they are subjected to tension, fluid pressure, or shear stress that exceeds their frictional resistance to sliding. Where such disturbances are small in relation to the dimensions of a structure in a rock, their effect is to alter the mechanical properties of the rock mass, but this mass may in some cases still be treated as a continuum. Where these disturbances have significant dimensions, they must be treated as part of the structure or as a boundary.

The loads applied to a rock mass are generally due to gravity, and compressive stresses are encountered more often than not. Under these circumstances, the most important factor in connection with the properties and continuity of a rock mass is the friction between surfaces of cracks and joints of all sizes in the rock. If conditions are such that sliding is not possible on any surfaces, the system may be treated to a good approximation as a continuum of rock, with the properties of the average test specimen. If sliding is possible on any surface, the system must be treated as a system of discrete elements separated by these surfaces, with frictional boundary conditions over them.

2 Analysis of stress and strain

2.1 Introduction

In the study of the mechanics of particles, the fundamental kinematical variable that is used is the *position* of the body, and its two time derivatives, the *velocity* and the *acceleration*. The interaction of a given body with other bodies is quantified in terms of the *forces* that these other bodies exert on the first body. The effect that these forces have on the body is governed by Newton's law of motion, which states that the sum of the forces acting on a body is equal to the mass of the body times its acceleration. The condition for a body to be in equilibrium is that the sum of the external forces and moments acting on it must vanish.

These basic mechanical concepts such as position and force, as well as Newton's law of motion, also apply to extended, deformable bodies such as rock masses. However, these concepts must be altered somewhat, for various reasons. First, the fact that the force applied to a rock will, in general, vary from point to point, and will be distributed over the body must be taken into account. The idealization that forces act at localized points, which is typically used in the mechanics of particles, is not sufficiently general to apply to all problems encountered in rock mechanics. Hence, it is necessary to introduce the concept of *traction*, which is a force per unit area. As the traction generally varies with the orientation of the surface on which it acts, it is most conveniently represented with the aid of an entity known as the *stress tensor*.

Another fundamental difference between the mechanics of particles and deformable bodies such as rocks is that different parts of the rock may undergo different amounts of displacement. In general, it is the relative displacement of neighboring particles, rather than the absolute displacement of a particular particle, that can be equating in some way to the applied tractions. This can be seen from the fact that a rock sample can be moved *as a rigid body* from one location to another, after which the external forces acting on the rock can remain unaltered. Clearly, therefore, the displacement itself cannot be directly related to the applied loads. This relative displacement of nearby elements of the rock is quantified by an entity known as the *strain*.

The stress tensor is a symmetric second-order tensor, and many important properties of stress follow directly from those of second-order tensors. In the event that the relative displacements of all parts of the rock are small, the strain can also be represented by a second-order tensor called the *infinitesimal strain* *tensor*. A consequence of this fact is that much of the general theory of stresses applies also to the analysis of strains. The general theory of stress and strain is the topic of this chapter. Both of these theories can be developed without any reference to the specific properties of the material under consideration (i.e., the constitutive relationship between the stress and strain tensors). Hence, the discussion given in this chapter parallels, to a great extent, that which is given in many texts on elasticity, or solid mechanics in general. Among the many classic texts on elasticity that include detailed discussion of the material presented in this chapter are Love (1927), Sokolnikoff (1956), Filonenko-Borodich (1965), and Timoshenko and Goodier (1970). The chapter ends with a brief introduction to the theory of finite strains.

2.2 Definition of traction and stress

Consider a rock mass that is subject to some arbitrary set of loads. At any given point within this rock, we can imagine a plane slicing through the rock at some angle. Such a plane may in fact form an external boundary of the rock mass, or may represent a fictitious plane that is entirely internal to the rock. Figure 2.1a shows such a plane, along with a fixed (x, y) coordinate system. In particular, consider an element of that plane that has area A. Most aspects of the theory of stress (and strain) can be developed within a two-dimensional context, and extensions to three dimensions are in most cases straightforward. As most figures are easier to draw, and to interpret, in two dimensions than in three, much of the following discussion will be given first in two-dimensional form.

The plane shown in Fig. 2.1a can be uniquely identified by the unit vector that is perpendicular to its surface. The vector $\mathbf{n} = (n_x, n_y)$ is the *outward unit normal vector* to this plane: it has *unit* length, is *normal* to the plane, and points in the direction *away* from the rock mass. The components of this vector $\mathbf{n}$ are the direction cosines that the outward unit normal vector makes with the two coordinate axes. For example, a plane that is perpendicular to the x-axis would have $\mathbf{n} = (1, 0)$. As the length of any unit normal vector is unity, the Pythagorean theorem implies that $(n_x)^2 + (n_y)^2 = 1$. The unit normal vectors in the directions of the coordinate axes are often denoted by $\mathbf{e}_x = (1, 0)$ and $\mathbf{e}_y = (0, 1)$. The identification of a plane by its outward unit normal vector is employed frequently

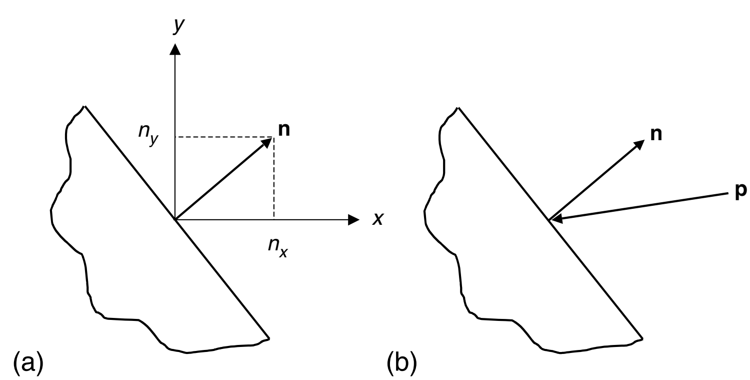

Fig. 2.1 Normal vector $\mathbf{n}$ and traction vector $\mathbf{p}$ acting on a surface.

in rock mechanics. It is important to remember that the vector **n** is perpendicular to the plane in question; it does not lie within that plane.

The action that the rock adjacent to the plane exerts on the rock that is "interior" to the plane can be represented by a resultant force **F**, which, like all forces, is a vector. The traction vector **p** is defined as the ratio of the resultant force **F** to the surface area A:

$$\mathbf{p}(\text{averaged over the area}) = \frac{1}{A}\mathbf{F}. \tag{2.1}$$

In order to define the traction that acts over a specific "point" in the rock, the area is now allowed to shrink down to a point, so that the magnitude A goes to zero. Following the convention often used in applied mathematics, the smallness of the area is indicated by the notation "dA," where the "d" denotes "differential," and likewise for the resultant force **F**. As the area shrinks down to a point, the traction at that point can then be defined by (Fig. 2.1b)

$$\mathbf{p}(\mathbf{x};\mathbf{n}) = \lim_{\mathrm{d}A\to 0}\frac{1}{\mathrm{d}A}\mathbf{dF}. \tag{2.2}$$

The notation $\mathbf{p}(\mathbf{x};\mathbf{n})$ denotes the traction vector at the point $\mathbf{x} \equiv (x, y, z)$, on a plane whose outward unit normal vector is **n**. In the following discussion, when the point **x** under consideration is either clear from the context, or immaterial to the particular discussion, the dependence of **p** on **x** will be suppressed in the notation.

At this point, it is necessary to introduce a sign convention that is inconsistent with the one used in most areas of mechanics, but which is nearly universal in the study of rocks and soils. The Cartesian component of the traction **p** in any given direction **r** is considered to be a positive number if the inner product (dot product) of **p** and a unit vector in the **r** direction is *negative*. One way to interpret this convention is that the traction is based on $-\mathbf{F}$, rather than **F**. The reason for utilizing this particular sign convention will become clear after the stresses are introduced.

It is apparent from the definition given in (2.2) that the traction is a vector, and therefore has two components in a two-dimensional system, and three components in a three-dimensional system. In general, this vector may vary from point to point, and is therefore a function of the location of the point in question. However, at any given point, the traction will also, in general, be different on different planes that pass through that point. In other words, the traction will also be a function of **n**, the outward unit normal vector. The fact that **p** is a function of two vectors, the position vector **x** and the outward unit normal vector **n**, is awkward. This difficulty is eliminated by appealing to the concept of *stress*, which was introduced in 1823 by the French civil engineer and mathematician Cauchy. The stress concept allows all possible traction vectors at a point to be represented by a single mathematical entity that does not explicitly depend on the unit normal of any particular plane. The price paid for this simplification, so to speak, is that the stress is not a vector, but rather a second-order tensor, which is a somewhat more complicated, and less familiar, mathematical object than is a vector.

Although there are an infinite number of different traction vectors at a point, corresponding to the infinity of possible planes passing through that point, all possible traction vectors can be found from knowledge of the traction vector on two mutually orthogonal planes (or three mutually orthogonal planes in three dimensions). To derive the relationship for the traction on an arbitrary plane, it is instructive to follow the arguments originally put forward by Cauchy. Consider a thin penny-shaped slab of rock having thickness h, and radius r (Fig. 2.2a). The outward unit normal vector on the right face of this slab is denoted by $\mathbf{n}$; the outward unit normal vector of the left face of the slab is therefore $-\mathbf{n}$. The total force acting on the face with outward unit normal vector $\mathbf{n}$ is equal to $\pi r^2\mathbf{p}(\mathbf{n})$, whereas the total force acting on the opposing face is $\pi r^2\mathbf{p}(-\mathbf{n})$. The total force acting on the outer rim of this penny-shaped slab will be given by an integral of the traction over the outer area, and will be proportional to $2\pi rh$, which is the surface area of the outer rim. Performing a force balance on this slab of rock yields

$$\pi r^2\mathbf{p}(\mathbf{n}) + \pi r^2\mathbf{p}(-\mathbf{n}) + 2\pi rh\mathbf{t} = 0, \tag{2.3}$$

where $\mathbf{t}$ is the mean traction over the outer rim. If the thickness h of this slab is allowed to vanish, this third term will become negligible, and the condition for equilibrium becomes

$$\mathbf{p}(-\mathbf{n}) = -\mathbf{p}(\mathbf{n}). \tag{2.4}$$

Equation (2.4), known as *Cauchy's first law*, essentially embodies a version of Newton's third law: if the material to the left of a given plane exerts a traction $\mathbf{p}$ on the material on the right, then the material on the right will exert a traction $-\mathbf{p}$ on the material to the left.

Now, consider a triangular slab of rock, as in Fig. 2.2b, with a uniform thickness w in the third (z) direction. Two faces of this slab have outward unit normal vectors that coincide with the negative x and y coordinate directions, respectively, whereas the third face has an outward unit normal vector of $\mathbf{n} = (n_x, n_y)$. The length of the face with outward unit normal vector $\mathbf{n}$ is taken to be h. The length of the face that has outward unit normal vector $\mathbf{n} = -\mathbf{e_x} = (-1, 0)$ is equal

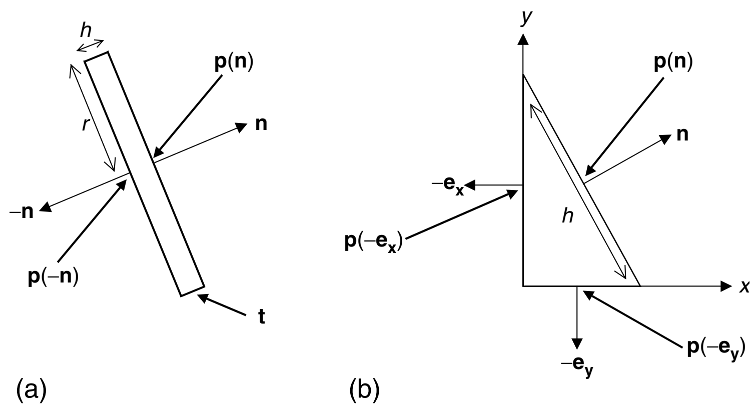

Fig. 2.2 (a) Thin slab used in derivation of Cauchy's first law; (b) triangular slab used in derivation of Cauchy's second law.

to hn_x, and so its area is hwn_x. The traction vector on this face is denoted by $\mathbf{p} = (-\mathbf{e_x})$, and so the total force acting on this face is $hwn_x\mathbf{p}(-\mathbf{e_x})$. Similar considerations show that the total force acting on the face with outward unit normal vector $-\mathbf{e_y}$ will be $hwn_y\mathbf{p}(-\mathbf{e_y})$. Hence, a force balance on this slab leads to

$$hwn_x\mathbf{p}(-\mathbf{e_x}) + hwn_y\mathbf{p}(-\mathbf{e_y}) + hw\mathbf{p}(\mathbf{n}) = 0. \tag{2.5}$$

Canceling out the common terms, and utilizing Cauchy's first law, (2.4), leads to *Cauchy's second law*:

$$\mathbf{p}(\mathbf{n}) = n_x\mathbf{p}(\mathbf{e_x}) + n_y\mathbf{p}(\mathbf{e_y}). \tag{2.6}$$

This result would remain unchanged if we consider the more general case in which a distributed *body force* acts on the tetrahedral-shaped element as in Fig. 2.2b. Whereas surface forces act over the outer surface of an element of rock, body forces act over the entire volume of the rock. The most obvious and common body force encountered in rock mechanics is that due to gravity, which has a magnitude of ρg (per unit volume), and is directed in the downward vertical direction. However, as will be shown in Chapter 7, gradients in temperature and pore fluid pressure also give rise to phenomena which have the same effect as distributed body forces. If there were a body-force density $\mathbf{b}$ per unit volume of rock, a total body force of $(1/2)h^2wn_xn_y\mathbf{b}$ would have to be added to the force balance in (2.5). If we divide through by h, and then let the size of the element shrink to zero (i.e., $h \to 0$), the body force term would drop out and $\mathbf{b}$ would not appear in the final result (2.6).

It is now convenient to recall that each traction is a vector, and therefore (in two dimensions) will have two components, one in each of the coordinate directions. The components of a traction vector such as $\mathbf{p}(\mathbf{e_x})$ are denoted using two indices – the first to indicate the direction of the outward unit normal vector and the second to indicate the component of the traction vector:

$$\mathbf{p}(\mathbf{e_x}) = \begin{bmatrix} \tau_{xx} \\ \tau_{xy} \end{bmatrix} = [\tau_{xx} \quad \tau_{xy}]^{\mathrm{T}}, \tag{2.7}$$

where we adhere to the algebraic convention that a vector is written as a column, and is therefore equivalent to the *transpose* of a row vector. Equation (2.6) can therefore be written in matrix form as

$$\mathbf{p}(\mathbf{n}) = n_x\begin{bmatrix} \tau_{xx} \\ \tau_{xy} \end{bmatrix} + n_y\begin{bmatrix} \tau_{yx} \\ \tau_{yy} \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{bmatrix}\begin{bmatrix} n_x \\ n_y \end{bmatrix}. \tag{2.8}$$

In the first expression on the right in (2.8), n_x and n_y are treated as scalars that multiply the two traction vectors; in the second expression, the formalism of matrix multiplication is used. As the two components of the vector $p(\mathbf{n})$ are $p_x(\mathbf{n})$ and $p_y(\mathbf{n})$, (2.8) can be written in component form as

$$p_x(\mathbf{n}) = \tau_{xx}n_x + \tau_{yx}n_y, \tag{2.9}$$

$$p_y(\mathbf{n}) = \tau_{xy}n_x + \tau_{yy}n_y. \tag{2.10}$$

If we use the standard matrix algebra convention that the first subscript of a matrix component denotes the row, and the second subscript denotes the column, the matrix appearing in (2.8) is seen to actually be the *transpose* of the stress matrix, in which case we can rewrite (2.8) as

$$\mathbf{p}(\mathbf{n}) = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} n_x \\ n_y \end{bmatrix}, \tag{2.11}$$

where the matrix that appears in (2.11), without the transpose operator, is the stress matrix, $\boldsymbol{\tau}$. Equation (2.11) can be written in direct matrix notation as

$$\mathbf{p} = \boldsymbol{\tau}^{\mathrm{T}}\mathbf{n}, \tag{2.12}$$

where $\mathbf{n}$ is a unit normal vector, $\mathbf{p}$ is the traction vector on the plane whose outward unit normal vector is $\mathbf{n}$, and $\boldsymbol{\tau}$ is the stress matrix, or *stress tensor*. In two dimensions the stress tensor has four components; in three dimensions it has nine. Equation (2.12) gives the traction on an arbitrarily oriented plane in terms of the stress matrix, relative to some fixed orthogonal coordinate system, and the direction cosines between the outward unit normal vector to the plane and the two coordinate axes. Note that a tensor can be written as a matrix, which is merely a rectangular array of numbers. However, a tensor has specific mathematical properties that are not necessarily shared by an arbitrary matrix-like collection of numbers. These properties relate to the manner in which the components of a tensor transform when the coordinate system is changed; these transformation laws are discussed in more detail in §2.3. The rows of the matrix that represents $\boldsymbol{\tau}$ are the traction vectors along faces whose outward unit normal vectors lie along the coordinate axes. In other words, the first row of $\boldsymbol{\tau}$ is $\mathbf{p}(\mathbf{e_x})$, the second row is $\mathbf{p}(\mathbf{e_y})$, etc.

The physical significance of the stress tensor is traditionally illustrated by the schematic diagram shown in Fig. 2.3a. Consider a two-dimensional square element of rock, whose faces are each perpendicular to one of the two coordinate axes. The traction vector that acts on the face whose outward unit normal vector is in the x direction has components (τ_{xx}, τ_{xy}). Each of these two components can be considered as a vector in its own right; they are indicated in Fig. 2.3a as arrows whose lines of action pass through the center of the face whose outward unit normal vector is $\mathbf{e_x}$. As the traction components are considered positive if they are oriented in the directions opposite to the outward unit normal vector, we see that the traction τ_{xx} is a positive number if it is *compressive*. Compressive stresses are much more common in rock mechanics than are tensile stresses. For example, the stresses in a rock mass that are due to the weight of the overlying rock are compressive. In most other areas of mechanics, tensile stresses are considered positive, and compressive stresses are reckoned to be negative. The opposite sign convention is traditionally used in rock (and soil) mechanics in order to avoid the frequent occurrence of negative signs in calculations involving stresses.

Many different notations have been used to denote the components of the stress tensor. We will mainly adhere to the notation introduced above, which has

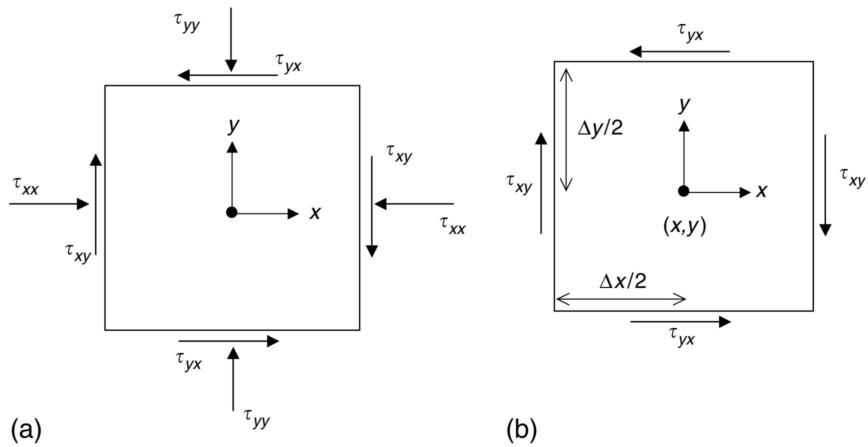

Fig. 2.3 (a) Stress components acting on a small square element. (b) Balance of angular momentum on this element shows that the stress tensor must be symmetric.

been used, for example, by Sokolnikoff (1956). Some authors use σ instead of τ as the basic symbol, but utilize the same subscripting convention. Many rock and soil mechanics treatments, including earlier editions of this book, denote shear stresses by τ_{xy}, etc., but denote normal stresses by, for example, σ_x rather than τ_{xx}. This notation, which has also been used by Timoshenko and Goodier (1970), has the advantage of clearly indicating the distinction between normal and shear components of the stress, which have very different physical effects, particularly when acting on fracture planes or other planes of weakness (Chapter 3). However, the $\{\sigma, \tau\}$ notation does not reflect the fact that the normal and shear components of the stress are both components of a single mathematical object known as the stress tensor. Furthermore, many of the equations in rock mechanics take on a simpler and more symmetric form if written in terms of a notation in which all stress components are written using the same symbol. However, a version of the Timoshenko and Goodier notation will occasionally be used in this book when discussing the traction acting on a specific plane. In such cases, for reasons of simplicity (so as to avoid the need for subscripts), it will be convenient to denote the normal stress by σ, and the shear stress by τ. Many classic texts on elasticity, such as Love (1927) and Filonenko-Borodich (1965), utilize the notation introduced by Kirchhoff in which τ_{xy} is denoted by X_y, etc. Green and Zerna (1954) use the notation suggested by Todhunter and Pearson (1886), in which τ_{xy} is denoted by $\widehat{xy}$, etc.

Equation (2.12) is usually written without the transpose sign, although strictly speaking the transpose is needed. The reason that it is allowable to write $\mathbf{p} = \boldsymbol{\tau}\mathbf{n}$ in place of $\mathbf{p} = \boldsymbol{\tau}^{\mathrm{T}}\mathbf{n}$ is that the stress matrix is in fact always *symmetric*, so that $\tau_{xy} = \tau_{yx}$, in which case $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{T}}$. This property of the stress tensor is of great importance, if for no other reason than that it reduces the number of stress components that must be measured or calculated from four to three in two dimensions, and from nine to six in three dimensions. The symmetry of the stress tensor can be proven by appealing to the law of conservation of angular momentum. For simplicity, consider a rock subject to a state of stress that does not vary from point to point. If we draw a free-body diagram for a small rectangular element of rock, centered at point (x, y), the traction components acting on the four faces are shown in Fig. 2.3b. The length of the element is Δx in the x direction, Δy in the y direction, and Δz in the z direction (into the page). In order for this element of rock to be in equilibrium, the sum of all the moments about any point, such as (x, y), must be zero. Consider first the tractions that act on the right face of the element. The force vector represented by this traction is found by multiplying the traction by the area of that face, which is $\Delta y \Delta z$. The x-component of this force is therefore $\tau_{xx} \Delta y \Delta z$. However, the resultant of this force acts through the point (x, y), and therefore contributes no moment about that point. The y-component of this traction is τ_{xy}, and the net force associated with it is $\tau_{xy} \Delta y \Delta z$. The moment arm of this force is $\Delta x/2$, so that the total clockwise moment about the z-axis, through the point (x, y), is $\tau_{xy} \Delta x \Delta y \Delta z/2$. Adding up the four moments that are contributed by the four shear stresses yields

$$\tau_{xy} \Delta x \Delta y \Delta z/2 + \tau_{xy} \Delta y \Delta x \Delta z/2 - \tau_{yx} \Delta x \Delta y \Delta z/2 - \tau_{yx} \Delta y \Delta x \Delta z/2 = 0. \tag{2.13}$$

Canceling out the terms $\Delta x \Delta y \Delta z/2$ leads to the result

$$\tau_{xy} = \tau_{yx}. \tag{2.14}$$

In three dimensions, a similar analysis leads to the relations $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$. This result should be interpreted as stating that at any specific point (x, y, z), the stress component $\tau_{xy}(x, y, z)$ is equal in magnitude and sign to the stress component $\tau_{yx}(x, y, z)$. There is in general no reason for the conjugate shear stresses *at different points* to be equal to each other.

Although the derivation presented above assumes that the stresses do not vary from point to point, and that the element of rock is in static equilibrium, the result is actually completely general. The reason for this is related to the fact that the result applies at each infinitesimal "point" in the rock. If we had accounted for the variations of the stress components with position, these terms would contribute moments that are of higher order in Δx and Δy. Dividing through the moment balance equation by $\Delta x \Delta y \Delta z$, and then taking the limit as Δx and Δy go to zero, would cause these additional terms to drop out, leading to (2.14). The same would occur if we considered the more general situation in which the element were not in static equilibrium, but rather was rotating. In this situation, the sum of the moments would be equal to the moment of inertia of the element about the z-axis through the point (x, y), which is $\rho \Delta x \Delta y \Delta z[(\Delta x)^2 + (\Delta y)^2]/12$, where ρ is the density of the rock, multiplied by the angular acceleration, $\dot{\omega}$. Hence, the generalization of (2.14) would be

$$\tau_{xy} \Delta x \Delta y \Delta z - \tau_{yx} \Delta x \Delta y \Delta z = \rho \Delta x \Delta y \Delta z[(\Delta x)^2 + (\Delta y)^2]\dot{\omega}/12. \tag{2.15}$$

Dividing through by $\Delta x \Delta y \Delta z$, and then taking the limit as the element shrinks down to the point (x, y), leads again to (2.14).

The symmetry of the stress tensor is therefore a general result. However, it is worth bearing in mind that although τ_{xy} and τ_{yx} are numerically equal, they are in fact physically distinct stress components, and act on different faces of an element of rock. Although the identification of τ_{xy} with τ_{yx} is eventually made when solving the elasticity equations, it is usually preferable to maintain a distinction between τ_{xy} and τ_{yx} when writing out equations, or drawing schematic diagrams such as Fig. 2.3a. This helps to preserve as much symmetry as possible in the structure of the equations.

The symmetry of the stress tensor followed from the principle of conservation of angular momentum. The principle of conservation of *linear* momentum leads to three further equations that must be satisfied by the stresses. These equations, which are known as the *equations of stress equilibrium* and are derived in §5.5, control the rate at which the stresses vary in space. However, much useful information about the stress tensor can be derived prior to considering the implications of the equations of stress equilibrium. Of particular importance are the laws that govern the manner in which the stress components vary as the coordinate system is rotated. These laws are derived and discussed in §2.3 and §2.5.

2.3 Analysis of stress in two dimensions

Discussions of stress are algebraically simpler in two dimensions than in three. In most instances, no generality is lost by considering the two-dimensional case, as the extension to three dimensions is usually straightforward. Furthermore, many problems in rock mechanics are essentially two dimensional, in the sense that the stresses do not vary along one Cartesian coordinate. The most common examples of such problems are stresses around boreholes, or around long tunnels. Many other problems are idealized as being two dimensional so as to take advantage of the relative ease of solving two-dimensional elasticity problems as compared to three-dimensional problems. Hence, it is worthwhile to study the properties of two-dimensional stress tensors. Various properties of two-dimensional stress tensors will be examined in this section; their three-dimensional analogues will be discussed in §2.5.

In order to derive the laws that govern the transformation of stress components under a rotation of the coordinate system, we again consider a small triangular element of rock, as in Fig. 2.4. The outward unit normal vector to

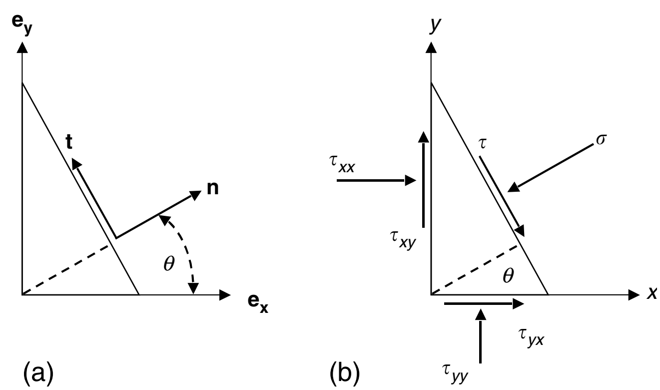

Fig. 2.4 Small triangular slab of rock used to derive the stress transformation equations.

the slanted face of the element is $\mathbf{n} = (n_x, n_y)$. We can construct another unit vector $\mathbf{t}$, perpendicular to $\mathbf{n}$, which lies along this face. Being of unit length, the components of $\mathbf{t}$ must satisfy the condition $\mathbf{t} \cdot \mathbf{t} = (t_x)^2 + (t_y)^2 = 1$. The orthogonality of $\mathbf{t}$ and $\mathbf{n}$ implies that $\mathbf{t} \cdot \mathbf{n} = t_x n_x + t_y n_y = 0$, which shows that $\mathbf{t} = \pm(n_y, -n_x)$. Finally, if we require that the pair of vectors $\{\mathbf{n}, \mathbf{t}\}$ have the same orientation relative to each other as do the pair $\{\mathbf{e_x}, \mathbf{e_y}\}$, the minus sign must be used, in which case $\mathbf{t} = (-n_y, n_x)$. This pair of vectors can be thought of as forming a new Cartesian coordinate system that is rotated from the original (x, y) system by a counterclockwise angle of $\theta = \arccos(n_x)$. According to (2.9) and (2.10), the components of the traction vector $\mathbf{p}(\mathbf{n})$, expressed in terms of the (x, y) coordinate system, are given by

$$p_x = \tau_{xx} n_x + \tau_{yx} n_y, \tag{2.16}$$

$$p_y = \tau_{xy} n_x + \tau_{yy} n_y. \tag{2.17}$$

In order to find the components of $\mathbf{p}$ relative to the $\{\mathbf{n}, \mathbf{t}\}$ coordinate system, we take the inner products of $\mathbf{p}$ with respect to $\mathbf{n}$ and $\mathbf{t}$, in turn. For example,

$$p_n = \mathbf{p} \cdot \mathbf{n} = p_x n_x + p_y n_y = \tau_{xx} n_x^2 + \tau_{yx} n_y n_x + \tau_{xy} n_x n_y + \tau_{yy} n_y^2. \tag{2.18}$$

Utilization of the symmetry property $\tau_{yx} = \tau_{yx}$ allows this to be written as

$$p_n = \tau_{xx} n_x^2 + 2\tau_{xy} n_x n_y + \tau_{yy} n_y^2. \tag{2.19}$$

Similarly, the tangential component of the traction vector on this face, which is given by $p_t = \mathbf{p} \cdot \mathbf{t}$, can be expressed as

$$p_t = (\tau_{yy} - \tau_{xx}) n_x n_y + \tau_{xy}(n_x^2 - n_y^2). \tag{2.20}$$

The two unit vectors $\{\mathbf{n}, \mathbf{t}\}$ can be thought of as defining a new coordinate system that is rotated by a counterclockwise angle θ from the old coordinate system. This interpretation is facilitated by denoting these two new unit vectors by $\{\mathbf{e_{x'}}, \mathbf{e_{y'}}\}$. Equations (2.19) and (2.20) are therefore seen to give the components of the traction vector on the plane whose outward unit vector is $\mathbf{e_{x'}}$, that is,

$$p_{x'}(\mathbf{e_{x'}}) \equiv \tau_{x'x'} = \tau_{xx} n_x^2 + 2\tau_{xy} n_x n_y + \tau_{yy} n_y^2, \tag{2.21}$$

$$p_{y'}(\mathbf{e_{x'}}) \equiv \tau_{x'y'} = (\tau_{yy} - \tau_{xx}) n_x n_y + \tau_{xy}(n_x^2 - n_y^2), \tag{2.22}$$

where, for clarity, we reemphasize that these components pertain to the traction on the plane with outward unit normal vector $\mathbf{e_{x'}}$. According to the discussion given in §2.2, these components can also be interpreted as the components of the stress tensor in the (x', y') coordinate system. Specifically, $p_{x'}(\mathbf{e_{x'}}) = \tau_{x'x'}$, and $p_{y'}(\mathbf{e_{x'}}) = \tau_{x'y'}$. The traction vector on the plane whose outward unit normal vector is $\mathbf{e_{y'}}$ can be found by a similar analysis; the results are

$$p_{y'}(\mathbf{e_{y'}}) \equiv \tau_{y'y'} = \tau_{xx} n_y^2 - 2\tau_{xy} n_x n_y + \tau_{yy} n_x^2, \tag{2.23}$$

$$p_{x'}(\mathbf{e_{y'}}) \equiv \tau_{y'x'} = (\tau_{yy} - \tau_{xx}) n_x n_y + \tau_{xy}(n_x^2 - n_y^2). \tag{2.24}$$

Note that $p_{x'}(\mathbf{e}_{\mathbf{y}'}) = \tau_{y'x'} = p_{y'}(\mathbf{e}_{\mathbf{x}'}) = \tau_{x'y'}$, as must necessarily be the case, due to the general property of symmetry of the stress tensor.

Another common notation used for the stress transformation equations in two dimensions can be obtained by recalling that the primed coordinate system is derived from the unprimed system by rotation through a counterclockwise angle of $\theta = \arccos(n_x)$. Furthermore, the components (n_x, n_y) of the unit normal vector $\mathbf{n}$ can be written as $(\cos\theta, \sin\theta)$. In terms of the angle of rotation, the stresses in the rotated coordinate system are

$$\tau_{x'x'} = \tau_{xx}\cos^2\theta + 2\tau_{xy}\sin\theta\cos\theta + \tau_{yy}\sin^2\theta, \tag{2.25}$$

$$\tau_{y'y'} = \tau_{xx}\sin^2\theta - 2\tau_{xy}\sin\theta\cos\theta + \tau_{yy}\cos^2\theta, \tag{2.26}$$

$$\tau_{x'y'} = (\tau_{yy} - \tau_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta). \tag{2.27}$$

This rotation operation can be represented by the rotation matrix $\mathbf{L}$, which has the defining properties that $\mathbf{L}^{\mathrm{T}}\mathbf{e}_{\mathbf{x}} = \mathbf{e}_{\mathbf{x}'}$, and $\mathbf{L}^{\mathrm{T}}\mathbf{e}_{\mathbf{y}} = \mathbf{e}_{\mathbf{y}'}$. In component form, relative to the (x, y) coordinate system, the two primed unit vectors are given by $\mathbf{e}_{\mathbf{x}'} = (\cos\theta, \sin\theta)$ and $\mathbf{e}_{\mathbf{y}'} = (-\sin\theta, \cos\theta)$. These two vectors therefore form the two columns of the matrix $\mathbf{L}^{\mathrm{T}}$ (Lang, 1971, p. 120), which is to say they form the *rows* of $\mathbf{L}$, that is,

$$\mathbf{L} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{2.28}$$

Using this rotation matrix, the transformation equations (2.25)–(2.27) can be written in the following matrix form:

$$\begin{bmatrix} \tau_{x'x'} & \tau_{x'y'} \\ \tau_{y'x'} & \tau_{y'y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{2.29}$$

which can also be expressed in direct matrix notation as

$$\boldsymbol{\tau}' = \mathbf{L}\boldsymbol{\tau}\mathbf{L}^{\mathrm{T}}. \tag{2.30}$$

The fact that the stresses transform according to (2.30) when the coordinate system is rotated is the defining property that makes the stress a *second-order tensor*. We note also that, using this direct matrix notation, the traction vector transforms according to $\mathbf{p}' = \mathbf{L}\mathbf{p}$. The appearance of *one* rotation matrix in this transformation law is the reason that vectors are referred to as *first-order tensors*.

The form of the stress transformation law given in (2.29) or (2.30) is convenient when considering a rotation of the coordinate system. However, from a more physically based viewpoint, it is pertinent to focus attention on the tractions that act on a given plane, such as the one shown in Fig. 2.4. The same equations are used in both situations, but their interpretation is slightly different. When focusing on a specific plane with unit normal vector $\mathbf{n}$, it is convenient to simplify the equations by utilizing the trigonometric identities $\cos^2\theta - \sin^2\theta = \cos 2\theta$, and $2\sin\theta\cos\theta = \sin 2\theta$. As long as attention is focused on a given plane, no confusion should arise if the normal stress acting on this plane is denoted by σ, and the shear stress is denoted by τ. After some algebraic manipulation, we arrive at the following equations for the normal and shear stresses acting on a plane whose outward unit normal vector is rotated counterclockwise from the x direction by an angle θ:

$$\sigma = \frac{1}{2}(\tau_{xx} + \tau_{yy}) + \frac{1}{2}(\tau_{xx} - \tau_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta, \tag{2.31}$$

$$\tau = \frac{1}{2}(\tau_{yy} - \tau_{xx})\sin 2\theta + \tau_{xy}\cos 2\theta. \tag{2.32}$$

The variation of σ and τ with the angle of rotation is illustrated in Fig. 2.5, for the case where $\{\tau_{xx} = 4, \tau_{yy} = 2, \tau_{xy} = 1\}$.

An interesting question to pose is whether or not there are planes on which the shear stress vanishes, and where the stress therefore has purely a normal component. The answer follows directly from setting $\tau = 0$ in (2.32), and solving for

$$\tan 2\theta = \frac{2\tau_{xy}}{\tau_{xx} - \tau_{yy}}. \tag{2.33}$$

If $\tau_{xy} = 0$, then the plane with $\mathbf{n} = \mathbf{e_x}$ is already a shear-free plane, and (2.33) gives the result $\theta = 0$. In general, however, whatever the values of $\{\tau_{xy}, \tau_{xy}, \tau_{yy}\}$, there will always be two roots of (2.33) in the range $0 \le 2\theta < 2\pi$, and these roots will differ by π. Hence, there will be two values of θ that satisfy (2.33), differing by $\pi/2$, and lying in the range $0 \le \theta < \pi$; this situation will be discussed in more detail below. For now, note that if θ is defined by (2.33), it follows from elementary trigonometry that

$$\sin 2\theta = \pm[1 + \cos^2 2\theta]^{-1/2} = \pm\tau_{xy}[\tau_{xy}^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2]^{-1/2}, \tag{2.34}$$

$$\cos 2\theta = \pm[1 + \tan^2 2\theta]^{-1/2} = \pm\frac{1}{2}(\tau_{xx} - \tau_{yy})[\tau_{xy}^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2]^{-1/2}, \tag{2.35}$$

in which case the normal stress is found from (2.31) to be given by

$$\sigma = \frac{1}{2}(\tau_{xx} - \tau_{yy}) \pm [\tau_{xy}^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2]^{-1/2}. \tag{2.36}$$

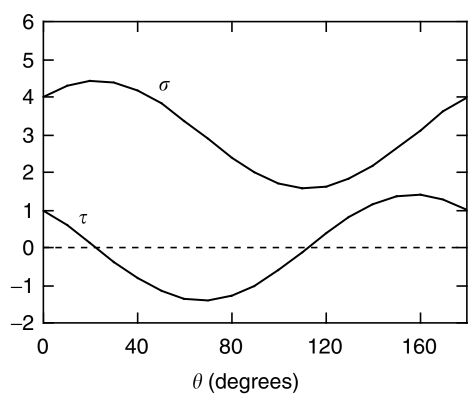

Fig. 2.5 Variation of normal and shear tractions with the angle θ (see Fig. 2.4a).

Equation (2.36) defines two normal stresses, σ_1 and σ_2, that are known as the *principal normal stresses*, or simply the *principal stresses*. These stresses act on planes whose orientations relative to the (x, y) coordinate system are given by (2.33). It is customary to set $\sigma_1 \geq \sigma_2$, in which case the + sign in (2.36) is associated with σ_1, that is,

$$\sigma_1 = \frac{1}{2}(\tau_{xx} + \tau_{yy}) + [\tau_{xy}^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2]^{1/2}, \tag{2.37}$$

$$\sigma_2 = \frac{1}{2}(\tau_{xx} + \tau_{yy}) - [\tau_{xy}^2 + \frac{1}{4}(\tau_{xx} - \tau_{yy})^2]^{1/2}. \tag{2.38}$$

These two principal normal stresses not only have the distinction of acting on planes on which there is no shear, but are also the minimum and maximum normal stresses that act on *any* planes through the point in question. This can be proven by noting that

$$\frac{d\sigma}{d\theta} = -(\tau_{xx} - \tau_{yy}) \sin 2\theta + 2\tau_{xy} \cos 2\theta = -2\tau, \tag{2.39}$$

so that any plane on which τ vanishes is also a plane on which σ takes on a locally extreme value. This is apparent from Fig. 2.5, which also shows, for example, that the shear traction τ will take on its maximum and minimum values on two orthogonal planes whose normal vectors bisect the two directions of principal normal stress.

Although it is clear from (2.37) and (2.38) which of the two principal stresses is largest, the direction in which the major principal stress acts is not so clear, due to the fact that (2.33) has two physically distinct solutions, that differ by $\pi/2$. The correct choice for σ_1 is the angle that makes the normal stress a local maximum, rather than a local minimum. To determine the correct value we examine the second derivatives of σ with respect to θ. From (2.39),

$$\frac{d^2\sigma}{d\theta^2} = -2(\tau_{xx} - \tau_{yy}) \cos 2\theta - 4\tau_{xy} \sin 2\theta. \tag{2.40}$$

Using (2.40), along with (2.33), eventually leads to the following results (Chou and Pagano, 1992, p. 10):

$$\tau_{xx} > \tau_{yy} \quad \text{and} \quad \tau_{xy} > 0 \quad \Rightarrow \quad 0 < \theta_1 < 45^\circ, \tag{2.41}$$

$$\tau_{xx} < \tau_{yy} \quad \text{and} \quad \tau_{xy} > 0 \quad \Rightarrow \quad 45^\circ < \theta_1 < 90^\circ, \tag{2.42}$$

$$\tau_{xx} < \tau_{yy} \quad \text{and} \quad \tau_{xy} < 0 \quad \Rightarrow \quad 90^\circ < \theta_1 < 135^\circ, \tag{2.43}$$

$$\tau_{xx} > \tau_{yy} \quad \text{and} \quad \tau_{xy} < 0 \quad \Rightarrow \quad 135^\circ < \theta_1 < 180^\circ. \tag{2.44}$$

The principal stresses and principal directions can also be found by a different method, which can more readily be generalized to three dimensions. We start again by asking whether or not there are planes on which the traction vector is purely normal, with no shear component. On such planes, the traction vector will be parallel to the outward unit normal vector, and can therefore be expressed as $\mathbf{p} = \sigma\mathbf{n}$, where σ is some (as yet unknown) scalar. From (2.27) it is known that $\mathbf{p} = \boldsymbol{\tau}^T\mathbf{n}$, which, due to the symmetry of the stress tensor, can be written as $\mathbf{p} = \boldsymbol{\tau}\mathbf{n}$. Hence, any plane on which the traction is purely normal must satisfy the equation

$$\boldsymbol{\tau}\mathbf{n} = \sigma\mathbf{n}. \tag{2.45}$$

The left-hand side of (2.45) represents a matrix, $\boldsymbol{\tau}$, multiplying a vector, $\mathbf{n}$, whereas on the right-hand side the vector $\mathbf{n}$ is multiplied by a scalar, σ. If the 2×2 identity matrix is denoted by $\mathbf{I}$, then $\mathbf{n} = \mathbf{In}$, and (2.45) can be rewritten as

$$(\boldsymbol{\tau} - \sigma\mathbf{I})\mathbf{n} = 0. \tag{2.46}$$

Equation (2.46) can be recognized as a standard eigenvalue problem, in which σ is the *eigenvalue*, and $\mathbf{n}$ is the *eigenvector*. Much of the theory of stress follows immediately from the theory pertaining to eigenvectors and eigenvalues of a symmetric matrix. The main conclusions of this theory in an arbitrary number of dimensions N are (Lang, 1971) that there will always be N mutually orthogonal eigenvectors, each corresponding to a real eigenvalue σ, although the eigenvalues need not necessarily be distinct from each other. In the present case, the eigenvalues are the principal stresses, and the associated eigenvectors are the principal stress directions. These results, along with explicit expressions for the principal stresses and principal stress directions, can be derived from (2.46) without appealing to the general theory, however, as follows.

Equation (2.46) can be written in component form as

$$(\tau_{xx} - \sigma)n_x + \tau_{xy}n_y = 0, \tag{2.47}$$

$$\tau_{yx}n_x + (\tau_{yy} - \sigma)n_y = 0. \tag{2.48}$$

Using the standard procedure of Gaussian elimination, we multiply (2.47) by τ_{yx}, and multiply (2.48) by $(\tau_{xx} - \sigma)$, to arrive at

$$(\tau_{xx} - \sigma)\tau_{yx}n_x + \tau_{xy}\tau_{yx}n_y = 0, \tag{2.49}$$

$$(\tau_{xx} - \sigma)\tau_{yx}n_x + (\tau_{yy} - \sigma)(\tau_{xx} - \sigma)n_y = 0. \tag{2.50}$$

Subtraction of (2.49) from (2.50) yields

$$[\sigma^2 - (\tau_{xx} + \tau_{yy})\sigma + (\tau_{xx}\tau_{yy} - \tau_{xy}^2)]n_y = 0, \tag{2.51}$$

where use has been made of the relationship $\tau_{yx} = \tau_{xy}$. This equation will be satisfied if either the bracketed term vanishes, or if $n_y = 0$. In this latter case, we must have $n_x = 1$, since $\mathbf{n}$ is a unit vector. Equation (2.47) then shows that $\sigma = \tau_{xx}$, and (2.48) shows that $\tau_{xy} = 0$. This solution therefore corresponds to the special case in which the x direction is already a principal stress direction, and τ_{xx} is a principal stress. In general, this will not be the case, and we must proceed by setting the bracketed term to zero:

$$\sigma^2 - (\tau_{xx} + \tau_{yy})\sigma + (\tau_{xx}\tau_{yy} - \tau_{xy}^2) = 0. \tag{2.52}$$

The bracketed term in (2.51) is the *determinant* of the matrix $(\boldsymbol{\tau} - \sigma\mathbf{I})$, so (2.52) can be written symbolically as $\det(\boldsymbol{\tau} - \sigma\mathbf{I}) = 0$, which is the standard criterion for finding the eigenvalues of a matrix. This equation is a quadratic in σ, and will always have two roots, which will be functions of the two parenthesized coefficients that appear in (2.52). Before discussing these roots, we note that as the two principal stresses are scalars, their values should not depend on the coordinate system used. Therefore, the two coefficients $(\tau_{xx} + \tau_{yy})$ and $(\tau_{xx}\tau_{yy} - \tau_{xy}^2)$, must be independent of the coordinate system being used; this could also be shown more directly by adding (2.25) and (2.26). These two combinations of the stress components are known as *invariants*, and are discussed in more detail in a three-dimensional context in §2.8.

The quadratic formula shows that the two roots of (2.52) are given by the two values σ_1 and σ_2 from (2.37) and (2.38). If σ takes on one of these two values, (2.47) and (2.48) become linearly dependent. In this case, one of the two equations is redundant, and we can solve (2.47) to find

$$\tan\theta = \frac{n_y}{n_x} = \frac{2\tau_{xy}}{(\tau_{xx} - \tau_{yy}) \pm [4\tau_{xy}^2 + (\tau_{xx} - \tau_{yy})^2]^{1/2}}, \tag{2.53}$$

where the + sign corresponds to σ_1, and the − sign corresponds to σ_2. Using the trigonometric identity $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$, it can be shown that (2.53) is consistent with (2.33). These two directions, corresponding to the two orthogonal unit eigenvectors, will define a new coordinate system, rotated by an angle θ from the x direction, in which the shear stresses are zero. This coordinate system is often referred to as the *principal coordinate system*.

2.4 Graphical representations of stress in two dimensions

A simple graphical construction popularized by Mohr (1914) can be used to represent the state of stress at a point. Recall that (2.31) and (2.32) give expressions for the normal stress and shear stress acting on a plane whose unit normal direction is rotated from the x direction by a counterclockwise angle θ. Now imagine that we are using the principal coordinate system, in which the shear stresses are zero and the normal stresses are the two principal normal stresses. In this case we replace τ_{xx} with σ_1, replace τ_{yy} with σ_2, replace τ_{xy} with 0, and interpret θ as the angle of counterclockwise rotation from the direction of the maximum principal stress. We thereby arrive at the following equations that give the normal and shear stresses on a plane whose outward unit normal vector is rotated by θ from the first principal direction:

$$\sigma = \frac{(\sigma_1 + \sigma_2)}{2} + \frac{(\sigma_1 - \sigma_2)}{2}\cos 2\theta, \tag{2.54}$$

$$\tau = \frac{-(\sigma_1 - \sigma_2)}{2}\sin 2\theta. \tag{2.55}$$

These are the equations of a circle in the (σ, τ) plane, with its center at the point $\{\sigma = (\sigma_1 + \sigma_2)/2, \tau = 0\}$, and with radius $(\sigma_1 - \sigma_2)$. In contrast to the standard parameterization in which the angle θ is measured in the counterclockwise direction, this circle is parameterized in the clockwise direction, with angle 2θ. This becomes clear if we note that $\cos 2\theta$ can be replaced with $\cos(-2\theta)$ in (2.54),

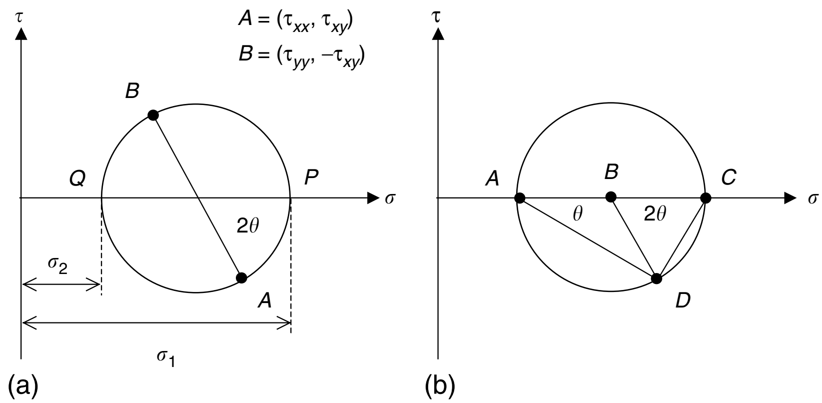

Fig. 2.6 Mohr's circle construction (see text for discussion).

and $-\sin 2\theta$ can be replaced with $\sin(-2\theta)$ in (2.55). A generic Mohr's circle is shown in Fig. 2.6a. A detailed discussion of the use of Mohr's circle in rock and soil mechanics is given by Parry (1995).

Many of the important properties of the two-dimensional stress tensor can be read directly off of the Mohr's circle. For example, at point P, when $\theta = 0$, there is no rotation from the σ_1 direction, and indeed Mohr's circle indicates that $(\sigma = \sigma_1, \tau = 0)$. Similarly, consider the plane for which $\theta = 90°$. This plane is rotated counterclockwise by 90° from the σ_1 direction, and therefore represents the σ_2 direction. This plane is represented on Mohr's circle by the point that is rotated clockwise by $2\theta = 180°$, which is point Q on Fig. 2.6a, where we find $(\sigma = \sigma_2, \tau = 0)$. This construction also clearly shows that the maximum shear stress has a magnitude equal to the radius of the Mohr's circle, and occurs on planes for which $2\theta = \pm 90°$, which is to say $\theta = \pm 45°$. These two planes bisect the two planes on which the principal normal stresses act, which are $\theta = 0°, 90°$.

Point A on the Mohr's circle in Fig. 2.6a shows the stresses acting on a generic plane whose unit normal vector is rotated by angle θ from the σ_1 direction. This direction can be denoted as the x direction, and these stresses can therefore be denoted by $(\sigma = \tau_{xx}, \tau = \tau_{xy})$. Now consider the plane that is rotated by an additional 90°. For this plane, the additional increment in 2θ is 180°, and the stresses are represented by the point B, which is located at the opposite end of a diameter of the circle from point A. This direction can be denoted as the y direction, in which case the x and y directions define an orthogonal coordinate system. However, the stresses at point B on Mohr's circle must be identified as $(\sigma = \tau_{yy}, \tau = -\tau_{yx})$. This is because it is implicit in (2.55) that the tangential direction is rotated 180° counterclockwise from the normal direction of the plane in question, which would then correspond to the $-x$ direction instead of the $+x$ direction.

It is also seen from Mohr's circle that the mean value of the two normal stresses, $(\tau_{xx} + \tau_{yy})/2$, is equal to the horizontal distance from the origin to the center of Mohr's circle, which is $(\sigma_1 + \sigma_2)/2$. This is another proof of the fact that the value of the mean normal stress is independent of the coordinate system used.

Mohr's circle can also be used to graphically determine the two principal stresses, and the orientations of the principal stress directions, given knowledge of the components of the stress tensor in some (x, y) coordinate system. We first plot the point (τ_{xx}, τ_{xy}) on the (σ, τ) plane, and note that these two stresses will be the normal and shear stresses on the plane whose outward unit normal vector is $\mathbf{e_x}$. This direction is rotated by some (as yet unknown) angle θ from the σ_1 direction. We next plot the stresses $(\tau_{yy}, -\tau_{yx})$ on the (σ, τ) plane, and note that these represent the stresses on the plane with outward unit normal vector $\mathbf{e_y}$. This direction is therefore rotated by an angle $\theta + 90°$ from the σ_1 direction. In accordance with the earlier discussion, the sign convention that is used for the shear stress on this second plane on a Mohr's diagram is opposite to that used when considering this as the second direction in an orthogonal coordinate system; hence, this second pair is plotted as $(\tau_{yy}, -\tau_{yx})$. As these two planes are rotated from one another by 90°, they will be separated by 180° on Mohr's circle; hence, the line joining these two points will be the diameter of Mohr's circle. Once this diameter is constructed, the circle can be drawn with a compass. The two points at which this circle intersects the σ-axis will be the two principal stresses, σ_1 and σ_2. The angle of rotation between the x direction and the σ_1 direction can also be read directly from this circle.

Mohr's circle can also be used to graphically find the orientation of the plane on which certain tractions act (Kuske and Robertson, 1974). Consider point D in Fig. 2.6b, at which the traction is given by (σ, τ). First note that $\angle DBA = \pi - \angle DBC$. Next, note that $\angle DAB$ and $\angle ADB$ are two equal angles of an isosceles triangle, the third angle of which is $\angle DBA$. It follows that $\angle DAB = \theta$. The chord AD therefore points in the direction of the outward unit normal vector to the plane in question. Since $\angle ADC$ is inscribed within a semicircle, we know that $\angle ADC = \pi/2$. Chords AD and DC are therefore perpendicular to each other, from which it follows that chord CD indicates the direction of the *plane* on which the tractions are (σ, τ). This construction is sometimes useful in aiding in the visualization of the tractions acting on various planes.

There are other geometrical constructions that have been devised to represent the state of stress at a point in a body. Most of these are less convenient than Mohr's circle, and to a great extent these graphical approaches, once very popular, have been superseded by algebraic methods. Nevertheless, we briefly mention Lamé's stress ellipsoid, which in two dimensions is a stress ellipse. To simplify the discussion, assume that we are using the principal coordinate system, in which case it follows from (2.9) and (2.10) that

$$p_1(\mathbf{n}) = \sigma_1 n_1 \quad \text{and} \quad p_2(\mathbf{n}) = \sigma_2 n_2, \tag{2.56}$$

where we have let $x \to 1, y \to 2$, and have noted that, by construction, $\tau_{12} = 0$. Since $\mathbf{n}$ is a unit vector, we see from (2.56) that

$$(p_1/\sigma_1)^2 + (p_2/\sigma_2)^2 = (n_1)^2 + (n_2)^2 = 1. \tag{2.57}$$

The point (p_1, p_2) therefore traces out an ellipse whose semimajor and semiminor axes are σ_1 and σ_2, respectively (Fig. 2.7). Each vector from the origin to a point

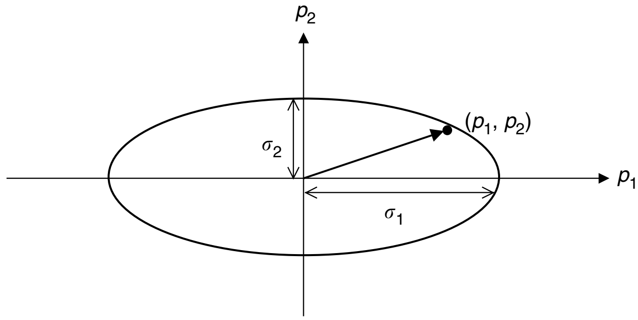

Fig. 2.7 Lamé's stress ellipse (see text for description).

on the ellipse represents a traction vector that acts on some plane passing through the point at which the principal stresses are σ_1 and σ_2. However, although the Lamé stress ellipse shows the various traction vectors that act on different planes, it does *not* indicate the plane on which the given traction acts. In general, only when the vector *OP* lies along one of the principal directions in Fig. 2.7 will the direction of the plane be apparent, since in these special cases the traction is known to be normal to the plane. In the more general case, the direction of the unit normal vector of the plane on which the traction is (p_1, p_2) can be found with the aid of the stress-director surface, which is defined by

$$(p_1^2/\sigma_1) + (p_2^2/\sigma_2) = \pm 1. \tag{2.58}$$

For the case which is most common in rock mechanics, in which both principal stresses are positive, the + sign must be used in (2.58), and the surface is an ellipse with axes $\sqrt{\sigma_1}$ and $\sqrt{\sigma_2}$. The outward unit normal vector of the plane on which the traction is (p_1, p_2) is then given by the tangent to the stress-director ellipse at the point where it intersects the stress ellipsoid (Chou and Pagano, 1992, p. 200). Proof of this assertion, and more details of this construction, can be found in Timoshenko and Goodier (1970) and Durelli et al. (1958).

One interesting fact that is more apparent from the Lamé construction than from Mohr's circle is that not only does the magnitude of the normal component of the stress take on stationary values in the principal directions, but the magnitude of the *total* traction vector also takes on stationary values in these directions. In particular, the maximum value of $|\mathbf{p}|$ is seen to be equal to σ_1, and occurs in the direction of the major principal stress.

Most of the manipulations and transformations described above are concerned with the values of the stress and traction at a given "point" in the rock. In general, the state of stress will vary from point to point. The equations that govern these variations are described in §5.5. The state of stress in a rock mass can either be estimated based on a solution (either numerical or analytical) of these equations (Chapter 8), or from stress measurements (Chapter 13). In order to completely specify the state of stress in a two-dimensional rock mass, it is necessary either to know the values of τ_{xx}, τ_{yy}, and τ_{xy} at each point in the body, or, alternatively, to know at each point the values of the two principal stresses σ_1 and σ_2, along with the angle of inclination between the x direction and, say, the σ_1 direction. Although it is difficult to display all of this information graphically, there are a number of simple graphical representations that are useful in giving a partial picture of a stress field. Among these are:

1 *Isobars*, which are curves along which the principal stress is constant. There are two sets of isobars, one for σ_1 and one for σ_2. A set of isobars for one of the principal stresses, say σ_1, must by definition form a nonintersecting set of curves. However, an isobar of σ_1 may intersect an isobar of σ_2.
2 *Isochromatics*, which are curves along which the maximum shear stress $(\sigma_1 - \sigma_2)/2$, is constant. These curves can be directly found using the methods of photoelasticity, which is described by Frocht (1941) and Durelli et al. (1958).
3 *Isopachs*, which are curves along which the mean normal stress $(\sigma_1 + \sigma_1)/2$ is constant. It is shown in §5.5 that this quantity satisfies Laplace's equation, which is the same equation that governs, for example, steady-state temperature distributions, or steady-state distributions of the electric field, in isotropic conducting bodies. Hence, the isopachs can be found from analogue methods that utilize electrically conducting paper that is cut to the same shape as the rock mass under investigation. This procedure is discussed by Durelli et al. (1958).
4 *Isostatics*, or *stress trajectories*, are a system of curves which are at each point tangent to the principal axes of the stress. As the two principal axes are always orthogonal, the two sets of isostatic curves are mutually orthogonal. Since a free surface is always a principal plane (as it has no shear stress acting on it), an isostatic curve will intersect a free surface at a right angle to it.
5 *Isoclinics*, which are curves on which the principal axes make a constant angle with a given fixed reference direction. These curves can also be obtained by photoelastic methods.
6 *Slip lines*, which are curves on which the shear stress is a maximum. As the maximum shear stress at any point is always in a direction that bisects the two directions of principal normal stresses, these lines form an orthogonal grid.

2.5 Stresses in three dimensions

The theory of stresses in three dimensions is in general a straightforward extension of the two-dimensional theory. A generic plane in three dimensions will have a unit normal vector $\mathbf{n} = (n_x, n_y, n_z)$. The components of this vector satisfy the normalization condition $(n_x)^2 + (n_y)^2 + (n_z)^2 = 1$. A three-dimensional version of the argument accompanying Fig. 2.2b leads to the following generalization of (2.6):

$$\mathbf{p}(\mathbf{n}) = n_x \mathbf{p}(\mathbf{e_x}) + n_y \mathbf{p}(\mathbf{e_y}) + n_z \mathbf{p}(\mathbf{e_z}). \tag{2.59}$$

The components of the three traction vectors that act on planes whose outward unit normals are in the three coordinate directions are denoted by

$$\mathbf{p}(\mathbf{e_x}) = [\tau_{xx} \quad \tau_{xy} \quad \tau_{xz}]^{\mathrm{T}}, \tag{2.60}$$

$$\mathbf{p}(\mathbf{e_y}) = [\tau_{yx} \quad \tau_{yy} \quad \tau_{yz}]^{\mathrm{T}}, \tag{2.61}$$

$$\mathbf{p}(\mathbf{e_z}) = [\tau_{zx} \quad \tau_{zy} \quad \tau_{zz}]^{\mathrm{T}}. \tag{2.62}$$

Substitution of (2.60)–(2.62) into (2.59) leads to

$$p_x(\mathbf{n}) = \tau_{xx}n_x + \tau_{yx}n_y + \tau_{zx}n_z, \tag{2.63}$$

$$p_y(\mathbf{n}) = \tau_{xy}n_x + \tau_{yy}n_y + \tau_{zy}n_z, \tag{2.64}$$

$$p_z(\mathbf{n}) = \tau_{xz}n_x + \tau_{yz}n_y + \tau_{zz}n_z, \tag{2.65}$$

which can be written in matrix form as $\mathbf{p}(\mathbf{n}) = \tau^{\mathrm{T}}\mathbf{n}$, that is,

$$\begin{bmatrix} p_x(\mathbf{n}) \\ p_y(\mathbf{n}) \\ p_z(\mathbf{n}) \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}. \tag{2.66}$$

The three-dimensional analogue of the argument illustrated by Fig. 2.3b would show that the conjugate terms in the three-dimensional stress tensor are equal, that is,

$$\tau_{yx} = \tau_{xy}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{zx} = \tau_{xz}. \tag{2.67}$$

Hence, (2.66) can also be written as $\mathbf{p}(\mathbf{n}) = \boldsymbol{\tau}\mathbf{n}$.

The question can again be asked as to whether or not there are planes on which the shear stresses vanish. On such planes, the traction vector will be parallel to the outward unit normal vector, and therefore can be written as $\mathbf{p} = \sigma\mathbf{n}$, where σ is some scalar. But as $\mathbf{p}(\mathbf{n}) = \boldsymbol{\tau}\mathbf{n}$, we have $\boldsymbol{\tau}\mathbf{n} = \sigma\mathbf{n} = \sigma\mathbf{I}\mathbf{n}$, and therefore again arrive at the eigenvalue problem $(\boldsymbol{\tau} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}$, §2.3 (2.46), that is,

$$\begin{bmatrix} \tau_{xx} - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} - \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.68}$$

From this point on, the development follows that for the two-dimensional theory. Although $(n_x, n_y, n_z) = (0, 0, 0)$ is obviously a solution to (2.68), it is inadmissible because it does not satisfy the condition that $\mathbf{n} \cdot \mathbf{n} = 1$. Admissible solutions can be found only if the determinant of the matrix $(\boldsymbol{\tau} - \sigma\mathbf{I})$ vanishes (Lang,1971). When the determinant is expanded out, it takes the form

$$\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0, \tag{2.69}$$

where

$$I_1 = \tau_{xx} + \tau_{yy} + \tau_{zz}, \tag{2.70}$$

$$I_2 = \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 - \tau_{xx}\tau_{yy} - \tau_{xx}\tau_{zz} - \tau_{yy}\tau_{zz}, \tag{2.71}$$

$$I_3 = \tau_{xx}\tau_{yy}\tau_{zz} + 2\tau_{xy}\tau_{xz}\tau_{yz} - \tau_{xx}\tau_{yz}^2 - \tau_{yy}\tau_{xz}^2 - \tau_{zz}\tau_{xy}^2. \tag{2.72}$$

The fact that the stress tensor is symmetric ensures that (2.69) has three *real* roots. These roots are conventionally labeled such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Each of these roots will correspond to an eigenvector that can be labeled as $\mathbf{n}^1 = (n_x^1, n_y^1, n_z^1)$, etc. Although, in general, eigenvectors are arbitrary to within a multiplicative